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NON-ITPFI DIFFEOMORPHISMS[†]

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ABSTRACT
We construct C^∞ diffeomorphisms of T^4 which give rise via the group measure space construction to factors which are not ITPFI. We extend the construction to arbitrary paracompact, connected manifolds of dimension $\geqq 6$.

Introduction

This paper extends results of [10–12, 14, 15, 19] and others concerning ergodic diffeomorphisms of C^∞ manifolds which do not preserve any σ-finite measure equivalent to the given smooth measure. In particular we are interested in the classification of ergodic group actions on a measure space generated by a single ergodic non-singular transformation up to orbit or weak equivalence (see §1 for definition). We describe the ratio set introduced by Krieger in [19], but we concentrate on type III_0 diffeomorphisms in this paper. (It has been shown that for each fixed $\lambda \in (0,1]$, all type III_λ transformations are weakly equivalent, but type III_0 transformations are highly non-unique, [3, 4, 19].)

The group measure space construction [25] gives a canonical method for associating to the action of an ergodic transformation on a Lebesgue space a von Neumann factor; weakly equivalent transformations give isomorphic factors (see [23, 28]). In this paper we construct type III_0 diffeomorphisms whose associated factors exhibit a special property, i.e., are non-ITPFI. Our construction is not on the algebraic level (we construct the diffeomorphisms, not the factors), although we use algebraic conditions for a factor to be non-ITPFI given by Connes and Woods [6, 7], and then apply Krieger's theorem [23] which includes the result that there is a one-to-one and onto correspondence between equivalence classes of ergodic measurable flows and flows of weights.

Krieger was the first to construct a non-ITPFI factor in 1970 [22]; it was

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Connes who proved that Krieger's factor was non-ITPFI [3]. Here we construct a C^∞ diffeomorphism of T^4 whose associated factor is non-ITPFI, or equivalently, a diffeomorphism which is not weakly equivalent to an odometer of product type (cf. §1). Katznelson has shown that there is a bijection between ITPFI factors and weak equivalence classes of C^2 diffeomorphisms of T^1 with irrational rotation numbers having unbounded continued fraction coefficients [15].

We have shown in [10] that every paracompact, connected manifold of dimension greater than or equal to three admits a smooth type III_0 diffeomorphism. All these examples seem to lie in the same (ITPFI) weak equivalence class. In this paper we give a different construction, from which we can obtain an uncountable family of non-weakly-equivalent type III_0 diffeomorphisms of T^3, and a non-ITPFI diffeomorphism of T^4. The method used is based on an example given in [8]. These diffeomorphisms have natural extensions to higher dimensional manifolds, which we give in §5.

In section 1 we introduce some necessary definitions and notation, and section 2 offers a short presentation of the flow associated to an ergodic automorphism; a more detailed version can be found in [8]. Section 3 gives a method for obtaining a C^∞ diffeomorphism of a manifold $X \times T^1$ whose associated flow is any prescribed measure-preserving C^∞ flow on a smooth manifold X. Sections 4 and 5 contain the examples mentioned above, which are obtained from the construction given in §3.

§1. Notation and definitions

Let $(X, \mathcal{S}, \mu)$ denote a Borel space where μ is a probability measure on $(X, \mathcal{S})$. We define f to be a non-singular ergodic transformation of $(X, \mathcal{S}, \mu)$ if $\mu \sim f_*\mu$ (where $f_*\mu(A) = \mu(f^{-1}A)$ for every $A \in \mathcal{S}$), and if every f-invariant set $B \in \mathcal{S}$ satisfies either $\mu(B) = 0$ or $\mu(B) = 1$. We define the set $\mathrm{Aut}(X, \mathcal{S}, \mu) = \{g : (X, \mathcal{S}, \mu) \circlearrowleft$ such that g is invertible, bimeasurable, and $g_*\mu \sim \mu\}$, and let $O_g(x) = \{g^n(x) : n \in \mathbf{Z}\}$. The *full group* of $g \in \mathrm{Aut}(X, \mathcal{S}, \mu)$ is defined by

$$[g] = \{h \in \mathrm{Aut}(X, \mathcal{S}, \mu) : h(x) \in O_g(x) \text{ for } \mu\text{-a.e. } x \in X\}.$$

DEFINITION 1.1. Two transformations $f, g \in \mathrm{Aut}(X, \mathcal{S}, \mu)$ are *weakly equivalent* or *orbit equivalent* if there exists a bimeasurable invertible map $\psi : X \to X$ with $\psi_*^{-1}\mu \sim \mu$ and $\psi(O_f(x)) = O_g(\psi(x))$ for μ-a.e. $x \in X$.

We now introduce an invariant of weak equivalence.

DEFINITION 1.2. Let $f \in \mathrm{Aut}(X, \mathcal{S}, \mu)$ be an ergodic transformation. A nonnegative real number t is said to be in the *ratio set of* f, $r^*(f)$, if for every Borel set $B \in \mathcal{S}$ with $\mu(B) > 0$, and for every $\varepsilon > 0$,

$$\mu\left(\bigcup_{n\in\mathbf{Z}}\left(B \cap f^n B \cap \left\{x \in X : \left|\frac{d\mu f^{-n}}{d\mu}(x) - t\right| < \varepsilon\right\}\right)\right) > 0.$$

Here $d\mu f^{-n}/d\mu$ denotes the Radon–Nikodym derivative of $f^n_*\mu$ with respect to μ. We set $r(f) = r^*(f)\backslash 0$. It has been shown that $r(f)$ is a closed subgroup of the multiplicative group of positive real numbers $\mathbf{R}^+$, and that f admits a σ-finite invariant measure equivalent to μ if and only if $r^*(f) = \{1\}$, [19]. If not, there are three possibilities:

(1) $r^*(f) = \{t \in \mathbf{R} : t \geqq 0\}$, in which case f is said to be of type III_1;

(2) $r^*(f) = \{0\} \cup \{\lambda^n : n \in \mathbf{Z}\}$ for $0 < \lambda < 1$; in this case f is said to be of type III_λ; or,

(3) $r^*(f) = \{0, 1\}$. Then f is of type III_0.

For each $\lambda \neq 0$, type III_λ automorphisms form a weak equivalence class, but type III_0 automorphisms are highly non-unique.

We define an *odometer of product type.*

Definition 1.3. Let $\{n_k\}_{k=1}^\infty$ be a sequence of positive integers, and set $X = \Pi_{k=1}^\infty\{0, 1, \cdots, n_k - 1\}$, with the product Borel structure. Define T on X by:

$$(Tx)_k = \begin{cases} 0 & \text{if } k < N(x), \\ x_k + 1 & \text{if } k = N(x), \\ x_k & \text{if } k > N(x), \end{cases}$$

where $N(x) = \inf\{k \geqq 1 : x_k \neq n_k - 1\}$. (In particular, $T(\{n_k - 1\})$ is the zero sequence.) Let ν_k be a probability measure on $\{0, 1, \cdots, n_k - 1\}$ such that the probability of every digit is positive and the product measure $\nu = \Pi\,\nu_k$ is non-atomic on X. It is not hard to check that ν is ergodic and quasi-invariant under T. By $\mathcal{O}(\{n_k\}, \{\nu_k\})$ we denote the odometer of product type defined by T on (X, ν). An automorphism f of a Lebesgue space $(X, \mathcal{S}, \mu)$ is of *product type* (or *ITPFI*) if it is weakly equivalent to some $\mathcal{O}(\{n_k\}, \{\nu_k\})$.

Remarks. (1) One important method used to study weak equivalence classes of systems $f \in \mathrm{Aut}(X, \mathcal{S}, \mu)$ is to study the crossed product algebras $W^*(L^\infty(X, \mu), f)$; i.e., the group measure space construction of von Neumann [25]. An ergodic transformation has a von Neumann factor associated to it in a canonical way and weakly equivalent transformations give isomorphic von Neumann factors. These factors are sometimes called Krieger factors. In particular, automorphisms of product type give factors W^* which are ITPFI;

that is, $W^* = \bigotimes_{k=1}^{\infty}(M_k, \phi_k)$ acts on the Hilbert space $H_k = \bigotimes_{k=1}^{\infty}(H_k, \Phi_k)$ where the M_k are type I_{n_k} factors acting on H_k, $2 \leqq n_k \leqq \infty$, and $\phi_k(x) = (\Phi_k, x\Phi_k)$ is a faithful state on M_k. For details see [1]. From now on we will refer to automorphisms of product type as ITPFI automorphisms.

(2) Katznelson has shown that every C^2 diffeomorphism of $T^1 = \mathbf{R}/\mathbf{Z}$ whose rotation number has unbounded continued fraction coefficients is ITPFI, and that every odometer of product type is weakly equivalent to a C^∞ diffeomorphism of T^1 [15].

§2. The flow associated to an ergodic automorphism

A one parameter group $\{U_s : -\infty < s < \infty\}$ of automorphisms of $(X, \mathcal{S}, \mu)$ is called a *measurable non-singular flow* if the map $(x, s) \mapsto U_s x$ from $X \times \mathbf{R}$ onto X is measurable. If $\psi : X \to \mathbf{R}$ is measurable and satisfies $\psi(U_s x) = \psi(x)$ for μ-a.e. $x \in X$ and every $-\infty < s < \infty$, then ψ is called a $\{U_s\}$-invariant function. If $\{U_s\}$ admits no non-constant invariant functions then we say that $\{U_s\}$ is an *ergodic flow*. Two flows $U_s : (X, S, \mu) \circlearrowleft$ and $U'_s : (X', S', \mu') \circlearrowleft$ are *isomorphic* if there exists an invertible bimeasurable map $\rho : X \to X'$ such that $\rho_*^{-1}\mu \sim \mu$ and satisfying $U'_s\rho(x) = \rho U_s(x)$ for μ-a.e. $x \in X$.

The following definitions and results come from [8].

Definition 2.1. Let $(X, \mathcal{S}, \mu)$ be a Lebesgue space with μ a probability measure and let ζ be an arbitrary partition of X. We denote by $\mathcal{B}(\zeta)$ the sub-σ-algebra of $\mathcal{S}$ consisting of all sets in $\mathcal{S}$ which are unions of elements of ζ. We define ζ to be a *measurable partition* of X if there exists a countable set of sets B_n, $n = 1, 2, \cdots$ in $\mathcal{B}(\mathcal{S})$ such that for any $C_1, C_2 \in \zeta$, $C_1 \neq C_2$, there exists an n such that either $C_1 \subset B_n$ and $C_2 \subset X \setminus B_n$, or $C_2 \subset B_n$ and $C_1 \subset X \setminus B_n$.

Let ζ be a measurable partition of X and π the natural surjection from X onto X/ζ, i.e., $\pi x = \pi x'$ if x and x' are in the same element $C(\zeta)$ of ζ. We define $\mathcal{S}_\zeta$ to be the σ-algebra consisting of all sets $E \in X/\zeta$ such that $\pi^{-1}E \in \mathcal{B}(\zeta)$. Let $\mu_\zeta(E) = \mu(\pi^{-1}E)$ for all E in $\mathcal{S}_\zeta$. If μ_ζ has no atomic parts, then $(X/\zeta, \mathcal{S}_\zeta, \mu_\zeta)$ is a Lebesgue space, called the quotient measure space of $(X, \mathcal{S}, \mu)$ with respect to ζ.

We now consider the $\mathbf{Z}$-action of an ergodic automorphism f on $(X, \mathcal{S}, \mu)$, i.e., $(n, x) \mapsto f^n x$ for every $n \in \mathbf{Z}$, $x \in X$. We define

$$S_f(x, t) = \left(fx, t - \log \frac{d\mu f^{-1}}{d\mu}(x)\right) \qquad \text{for every } (x, t) \in X \times \mathbf{R}.$$

By $(X \times \mathbf{R}, \mathcal{S} \times \mathcal{T}, \mu \otimes \lambda)$ we will denote the measure space obtained by forming the cartesian product of $(X, \mathcal{S}, \mu)$ and $(\mathbf{R}, \mathcal{T}, \lambda)$, where λ denotes Haar (Lebesgue) measure and the product σ-algebra is formed in the usual way.

DEFINITION 2.2. A map ϕ from $X \times \mathbf{R}$ onto a Lebesgue space $(Y, \mathcal{F}, \nu)$ is called a *factor map* with respect to S_f if it satisfies:

(1) $\phi^{-1}A \in \mathcal{S} \times \mathcal{T}$ if and only if $A \in \mathcal{F}$.

(2) $\mu(\phi^{-1}A) = 0$ if and only if $\nu(A) = 0$, $\forall A \in \mathcal{F}$.

(3) $\phi \circ S_f(x, t) = \phi(x, t)$ for a.e. $(x, t) \in X \times \mathbf{R}$.

(4) If $\eta : X \times \mathbf{R}$ is an S_f-invariant function, then there is a function $\bar{\eta} : Y \to \mathbf{R}$ such that $\eta(x, t) = \bar{\eta}(\phi(x, t))$ for $\mu \otimes \lambda$-a.e. $(x, t) \in X \times \mathbf{R}$.

The following lemma states that factor maps are unique up to isomorphism.

LEMMA 2.3. [8] *Let ϕ_1 and ϕ_2 be measurable maps from $(X \times \mathbf{R}, \mathcal{S} \times \mathcal{T}, \mu \otimes \lambda)$ onto Lebesgue spaces $(Y_1, \mathcal{F}_1, \nu_1)$ and $(Y_2, \mathcal{F}_2, \nu_2)$ respectively, satisfying $\nu_i(A_i) = 0$ if and only if $\mu \otimes \lambda(\phi_i^{-1}A_i) = 0$, $A_i \in \mathcal{F}_i$ for $i = 1, 2$. If for any measurable function $\bar{\eta}_2$ on Y_2 there exists a measurable function $\bar{\eta}_1$ on Y_1 satisfying:*

$$\bar{\eta}_2(\phi_2(x, t)) = \bar{\eta}_1(\phi_1(x, t)) \qquad \text{for a.e. } (x, t) \in X \times \mathbf{R},$$

and if for any measurable function $\bar{\eta}_1$ on Y_1 there exists a measurable function $\bar{\eta}_2$ on Y_2 satisfying the above equation, then there exists an isomorphism $\psi : (Y_1, \mathcal{F}_1, \nu_1) \to (Y_2, \mathcal{F}_2, \nu_2)$ satisfying $\psi(\phi_1(x, t)) = \phi_2(x, t)$ for a.e. $(x, t) \in X \times \mathbf{R}$.

Let $\zeta(f)$ denote the measurable partition which generates all S_f invariant sets, and let π_f denote the natural surjection from $X \times \mathbf{R}$ onto the measure space $X \times \mathbf{R}/\zeta(f)$. It is easy to see that π_f is a factor map with respect to S_f. We now define a flow on $X \times \mathbf{R}$ by $T_t(x, s) = (x, s + t)$ for every $(x, s) \in X \in \mathbf{R}$, and $-\infty < t < \infty$. Since S_f commutes with $\{T_t\}$ for all $t \in \mathbf{R}$, the image under π_f of $\{T_t\}$ is a flow on $(X \times \mathbf{R}/\zeta(f), \mathcal{S}_\zeta, \mu_\zeta)$ defined by $\tilde{T}_t(\pi_f(x, s)) = \pi_f(T_t(x, s))$ for a.e. $(x, s) \in X \times \mathbf{R}$. It has been proved that weakly equivalent transformations $f : (X, \mathcal{S}, \mu) \circlearrowleft$ and $f' : (X', S', \mu') \circlearrowleft$ give rise via the above construction to isomorphic flows, $\{\pi_f T_t\}$ and $\{\pi_{f'} T'_t\}$, and we call the isomorphism class of the flow *the flow associated to f*. An automorphism f is of type III_0 if and only if its associated flow is an aperiodic conservative ergodic flow [8, 23].

§3. Construction of a type III_0 diffeomorphism

In [10] a method was given for constructing smooth type III_0 diffeomorphisms of any paracompact manifold of dimension greater than or equal to three, as well as on T^2. We give a different method here for constructing type III_0 diffeomorphisms, based on a construction of (non-smooth) automorphisms given by Hamachi and Osikawa [8]. This method only provides examples in dimensions $\geqq 3$ but the advantage of this construction is the following: given any smooth measure preserving flow, $\{U_s\}$, of a C^∞ manifold, we can construct a C^∞ diffeomorphism whose associated flow is $\{U_s\}$.

In section 4 we will use this construction to prove the existence of non-ITPFI diffeomorphisms since Connes and Woods proved that ITPFI transformations must have certain types of associated flows [6, 7].

The main theorem of this section gives a method for constructing an ergodic type III_0 transformation of a manifold, which may not be smooth, but is weakly equivalent to a smooth diffeomorphism with a prescribed associated flow. We first need a lemma proved in [8] about type III_1 transformations.

DEFINITION 3.1. Let $f \in \mathrm{Aut}(X, \mathcal{S}, \mu)$ be ergodic. If there exists an ergodic subgroup H of $[f]$ and a σ-finite measure $\bar{\mu}$ which is H-invariant and equivalent to μ, we say that $\bar{\mu}$ is an f-admissible measure. If there exists a countable subset $\Gamma \subset \mathbf{R}^+$ such that for any $n \in \mathbf{Z}$, $d\bar{\mu}f^{-n}(x)/d\bar{\mu} \in \Gamma$ for a.e. $x \in X$, then $\bar{\mu}$ is *strictly admissible.*

LEMMA 3.2. *Every ergodic automorphism f of a Lebesgue space $(X, \mathcal{S}, \mu)$ of type* III_1 *admits a strictly admissible finite measure $\bar{\mu}$ and*

$$\Delta(\bar{\mu}, f) = \left\{ r \in \Gamma \,\middle|\, \exists n \in \mathbf{Z} \text{ s.t. } \mu\left\{x : \frac{d\bar{\mu}f^{-n}}{d\bar{\mu}}(x) = r\right\} > 0 \right\}$$

is a dense subgroup of $\mathbf{R}^+$.

In what follows, $(X, \mathcal{S}, \mu)$ will denote a paracompact C^∞ manifold, μ a smooth σ-finite measure, and $\mathcal{S}$ will denote the σ-algebra of Borel sets of X. Similarly, $(Y, \mathcal{F}, \nu)$ will denote a C^∞ manifold with ν a smooth measure.

THEOREM 3.3. *Let $f \in \mathrm{Diff}^\infty(X)$ be an ergodic diffeomorphism of X, and let $\{U_s\}$ be a C^∞ flow on $(Y, \mathcal{F}, \nu)$ which is aperiodic, ergodic, and measure preserving.*

Assume that f is of type III_1 *and that $\bar{\mu} \sim \mu$ is a strictly f-admissible measure, and define a transformation on $X \times Y$ by:*

$$F(x, y) = (fx, U_{\log(d\bar{\mu}f^{-1}(x)/d\bar{\mu})}(y)) \qquad \textit{for every } x \in X \textit{ and } y \in Y.$$

Then F is an ergodic transformation of $(X \times Y, \mathcal{S} \times \mathcal{F}, \mu \otimes \nu)$ which gives a type III_0 $\mathbf{Z}$*-action, and whose associated flow is $\{U_s\}$.*

PROOF. Let $\varphi : X \times Y \times \mathbf{R} \to Y$ be defined by $\varphi(x, y, t) = U_t(y)$ for every $x \in X$, $y \in Y$, $t \in \mathbf{R}$. Consider the transformation S_F defined as in §2:

$$S_F(x, y, t) = \left(fx,\ U_{\log(d\bar{\mu}f^{-1}(x)/d\bar{\mu})}(t),\ t - \log \frac{d\bar{\mu} \otimes \nu F^{-1}}{d\bar{\mu} \otimes \nu}(x, y)\right).$$

We claim that $\varphi \circ S_F = \varphi$, $\bar{\mu} \otimes \nu \otimes m$-a.e. (where m denotes Lebesgue measure on $\mathbf{R}$). The claim is true because for any $(x, y, t) \in X \times Y \times \mathbf{R}$,

$$\begin{aligned}\varphi \circ S_F(x, y, t) &= \varphi\left(fx,\ U_{\log(d\bar{\mu}f^{-1}(x)/d\bar{\mu})}(y),\ t - \log \frac{d\bar{\mu} \otimes \nu F^{-1}}{d\bar{\mu} \otimes \nu}(x, y)\right) \\ &= U_{t - \log(d\bar{\mu} \otimes \nu F^{-1}(x,y)/d\bar{\mu} \otimes \nu)}\left(U_{\log(d\bar{\mu}f^{-1}(x)/d\bar{\mu})}(y)\right) \\ &= U_{t - \log(d\bar{\mu} \otimes \nu F^{-1}(x,y)/d\bar{\mu} \otimes \nu) + \log(d\bar{\mu}f^{-1}(x)/d\bar{\mu})}(y).\end{aligned}$$

Now since U_s preserves ν, we have

$$\log \frac{d\bar{\mu} \otimes \nu F^{-1}}{d\bar{\mu} \otimes \nu}(x, y) = \log\left(\det\begin{pmatrix} Df^{-1}(x) & 0 \\ D_x U_l(x, y) & D_y U_l(x, y)\end{pmatrix}\right),$$

where

$$l = \log \frac{d\bar{\mu}f^{-1}}{d\bar{\mu}}(x),$$

so

$$\log \frac{d\bar{\mu} \otimes \nu F^{-1}}{d\bar{\mu} \otimes \nu}(x, y) = \log\left(\det\begin{pmatrix} Df^{-1}(x) & 0 \\ D_x U_l(x, y) & 1\end{pmatrix}\right) = \log \frac{d\bar{\mu}f^{-1}}{d\bar{\mu}}(x).$$

Thus $\varphi \circ S_F(x, y, t) = U_t(y) = \varphi(x, y, t)$ $\bar{\mu} \otimes \nu \otimes m$-a.e., proving the claim.

Our aim is to prove that φ is a factor map onto $X \times Y \times \mathbf{R}/\zeta(S_F) \cong Y$. In order to prove this we need to show that for any S_F-invariant function $\psi : X \times Y \times \mathbf{R} \to \mathbf{R}$ there is a function $\tilde{\psi}$ defined on Y such that $\psi(x, y, t) = \tilde{\psi}(\varphi(x, y, t)) = \tilde{\psi}(U_t(y))$ for a.e. $(x, y, t) \in X \times Y \times \mathbf{R}$.

Suppose that ψ is S_F-invariant. Consider all $h \in [f]$ such that

$$\frac{d\bar{\mu}h^{-1}}{d\bar{\mu}}(x) = 1 \qquad \text{for } \bar{\mu}\text{-a.e. } x \in X. \tag{3.1}$$

We know that there exist automorphisms h satisfying (3.1) by Lemma 3.2, since $\bar{\mu}$ is an f-admissible measure. Then for a.e. $(x, y, t) \in X \times Y \times \mathbf{R}$,

$$\psi(hx, y, t) = \psi \circ S_F(hx, y, t) \tag{3.2}$$

$$= \psi\left(f(hx),\ U_{\log(d\bar{\mu}f^{-1}(hx)/d\bar{\mu})}(y),\ t - \log \frac{d\bar{\mu}f^{-1}}{d\bar{\mu}}(hx)\right) \tag{3.3}$$

$$(3.4) \qquad = \psi\left(f(f^{n(x)}x), U_{\log(d\bar{\mu}f^{-1}(f^{n(x)}x)/d\bar{\mu})}(y), t - \log\frac{d\bar{\mu}f^{-1}}{d\bar{\mu}}(f^{n(x)}x)\right).$$

By (3.1) and the chain rule,

$$(3.5) \qquad = \psi\left(f^{n(x)+1}x, U_{\log(d\bar{\mu}f^{-(n(x)+1)}(x)/d\bar{\mu})}(y), t - \log\frac{d\bar{\mu}f^{-(n(x)+1)}}{d\bar{\mu}}(x)\right)$$

$$= \psi(x, y, t).$$

Since the group satisfyting (3.1) is ergodic, ψ must be $\bar{\mu}$-a.e. constant with respect to x, so ψ is a function of (y, t). Then we have:

$$(3.6) \qquad \psi\left(U_{\log(d\bar{\mu}f^{-1}(x)/d\bar{\mu})}(y), t - \log\frac{d\bar{\mu}f^{-1}}{d\bar{\mu}}(x)\right) = \psi(y, t)$$

for a.e. $(x, y, t) \in X \times Y \times \mathbf{R}$. Lemma 3.2 implies that the set

$$\left\{\log\frac{d\bar{\mu}f^{-n}}{d\bar{\mu}}(x) : n \in \mathbf{Z}\right\}$$

is dense in $\mathbf{R}$ for $\bar{\mu}$-a.e. $x \in X$, and since the flow $\{U_s\}$ is continuous, we have $\psi(U_s(y), t - s) = \psi(y, t)$ for $\nu \otimes m$-a.e. $(y, t) \in Y \times \mathbf{R}$, for every $-\infty < s < \infty$. In particular, setting $s = t$, we have for $\bar{\mu} \otimes \nu \otimes m$-a.e. $(x, y, s) \in X \times Y \times \mathbf{R}$,

$$\psi(x, y, s) = \tilde{\psi}(y, s) = \tilde{\psi}(U_s(y), 0) = \tilde{\psi}(\varphi(x, y, s)).$$

Therefore φ is a factor map from $X \times Y \times \mathbf{R}$ onto Y with respect to S_F and the associated flow of F is $\{U_s\}$.

As an easy corollary of Theorem 3.3 we obtain a diffeomorphism which is of type III_0.

Corollary. 3.4. *If we replace $\bar{\mu}$ in the statement of Theorem 3.3 with the given smooth measure μ on X and define*

$$G(x, y) = (fx, U_{\log(d\mu f^{-1}(x)/d\mu)}(y)) \qquad \textit{for every } (x, y) \in X \times Y,$$

then G is a diffeomorphism which is weakly equivalent to F.

Proof. To see that G is a diffeomorphism, we remark first that $(x, y) \mapsto (fx, U_s(y))$ is a diffeomorphism for each $s \in \mathbf{R}$, and that

$$G(x, y) = (fx, U_{\log(d\mu f^{-1}(x)/d\mu)}(y))$$

is C^∞.

It is not difficult to see that G^{-1} exists and is defined by:

$$(3.7)\qquad (x, y) \mapsto (f^{-1}x, U_{\log(d\mu f(x)/d\mu)}(y)),$$

which is also C^∞. This proves that G is a diffeomorphism. (To check (3.7), we verify that $G \circ G^{-1}(x, y) = (x, y)$ as follows:

$$(3.8)\qquad G \circ G^{-1}(x, y) = G(f^{-1}x, U_{\log(d\mu f(x)/d\mu)}(y))$$

$$(3.9)\qquad = (f \circ f^{-1}x, U_{\log(d\mu f^{-1}(f^{-1}x)/d\mu)} \circ U_{\log(d\mu f(x)/d\mu)}(y))$$

$$(3.10)\qquad = (x, U_{\log(d\mu f^{-1}(f^{-1}x)/d\mu)+\log(d\mu f(x)/d\mu)}(y))$$

$$(3.11)\qquad = (x, y).$$

We obtain (3.11) from (3.10) since

$$0 = \log \frac{d\mu}{d\mu}(x) = \log \frac{d\mu(\mathrm{id})^{-1}}{d\mu}(x) = \log \frac{d\mu(f \circ f^{-1})^{-1}}{d\mu}(x)$$

$$= \log\left[\frac{d\mu f^{-1}}{d\mu}(f^{-1}x) \cdot \frac{d\mu f}{d\mu}(x)\right]$$

$$= \log \frac{d\mu f^{-1}}{d\mu}(f^{-1}x) + \log \frac{d\mu f}{d\mu}(x),$$

recalling that $d\mu f/d\mu$ denotes the Radon–Nikodym derivative of $f_*^{-1}\mu$ with respect to μ.

Similarly, we can show that $G^{-1} \circ G(x, y) = (x, y)$ for every $(x, y) \in X \times Y$.)

To show that G is weakly equivalent to F, we exhibit a measurable isomorphism which takes orbits of F to orbits of G. We define $H : X \times Y \to X \times Y$ by:

$$(3.12)\qquad H(x, y) = (x, U_{\log(d\mu(x)/d\bar{\mu})}(y))$$

for all $(x, y) \in X \times Y$. It is not difficult to see that H is measurable, invertible, and leaves the measure $\mu \otimes \nu$ on $X \times Y$ quasi-invariant. We claim that $H \circ F = G \circ H$ $\mu \otimes \nu$-a.e. To prove the claim, (3.12) implies:

$$(3.13)\qquad H \circ F(x, y) = H(fx, U_{\log(d\bar{\mu} f^{-1}(x)/d\bar{\mu})}(y))$$

$$(3.14)\qquad = (fx, U_{\log(d\mu(fx)/d\bar{\mu})+\log(d\bar{\mu} f^{-1}(x)/d\bar{\mu})}(y))$$

$$(3.15)\qquad = (fx, U_{\log(d\mu f^{-1}(x)/d\bar{\mu})}(y))$$

for $\mu \otimes \nu$-a.e. $(x, y) \in X \times Y$. Statements (3.14) and (3.15) are equal by an application of the chain rule.

Similarly,

$$G \circ H(x, y) = G(x, U_{\log(d\mu(x)/d\bar{\mu})}(y)) \tag{3.16}$$

$$= (fx, U_{\log(d\mu f^{-1}(x)/d\mu) + \log(d\mu(x)/d\bar{\mu})}(y)) \tag{3.17}$$

$$= (fx, U_{\log(d\mu f^{-1}(x)/d\bar{\mu})}(y)) \tag{3.18}$$

$$= H \circ F(x, y) \qquad \mu \otimes \nu\text{-a.e. by (3.13)–(3.15).}$$

This concludes the proof of the corollary.

REMARK 3.5. In [1] Araki and Woods constructed uncountably many non-isomorphic type III_0 factors. In [21] Krieger constructed an uncountable family of non-weakly-equivalent ergodic automorphisms of type III_0. Here we construct an uncountable family of non-weakly-equivalent type III_0 diffeomorphisms of T^3. We define the family, denoted G_λ, $0 < \lambda < 1$, as follows. Let $f \in \mathrm{Diff}^\infty(T^1)$ be of type III_1, and let $g_\lambda \in \mathrm{Diff}^\infty(T^1)$ be of type III_λ. These diffeomorphisms exist by [14]. Let U_s^λ denote the suspension flow of g_λ; i.e., the flow induced by $U_s(y, z) = (y, z + s)$ $\forall y \in T^1$, $z \in \mathbf{R}$, $s \in \mathbf{R}$ on the space $T^1 \times \mathbf{R}/(y, z) \sim (g_\lambda^n y, z + n)$ for all $y \in T^1$, $z \in \mathbf{R}$, $n \in \mathbf{Z}$. This defines an aperiodic, conservative, ergodic flow on T^2, which we call U_s^λ.

For $(x, y, z) \in T^3$, we define:

$$G_\lambda(x, y, z) = (fx, U^\lambda_{\log(dmf^{-1}(x)/dm)}(y, z)),$$

using m to denote Lebesgue measure on T^1. By Theorem 3.3 and Corollary 3.4 it follows that G_λ is of type III_0 with U_s^λ as its associated ergodic flow. Since g_λ is not weakly equivalent to g_β if $\lambda \neq \beta$, then U_s^λ is not isomorphic to U_s^β; hence G_λ and G_β cannot be weakly equivalent.

§4. Non-ITPFI diffeomorphisms

In this section we use results of Connes and Woods which give conditions for Krieger factors (cf. §1) to be non-ITPFI (see [6, 7, 28]). Combining these results with Krieger's theorem which gives an isomorphism between aperiodic, conservative, ergodic flows and flows of weights on type III_0 Krieger factors allows us to obtain non-ITPFI diffeomorphisms. In fact the flow associated to an ergodic group action of f on $(X, \mathcal{S}, \mu)$ is the same (up to isomorphism) as the flow of weights obtained from the Krieger factor $W^*(L^\infty(X, \mu), f)$. (See [27] for a good exposition of this point.)

We begin with a definition of a property which is stronger than ergodicity.

DEFINITION 4.1. [7] Let $(X, \mathcal{S}, \mu)$ be a Lebesgue space and let $\alpha : G \to \mathrm{Aut}(X, \mathcal{S}, \mu)$ be a homomorphism from a locally compact group G to the group of automorphisms of $(X, \mathcal{S}, \mu)$. We say that α is *approximately transitive* if given $\varepsilon > 0$ and $h_1, \cdots, h_r \in L^1_+(X, \mu)$, there exists $h \in L^1_+(X, \mu)$ and $\gamma_1, \cdots, \gamma_r \in L^1_+(G, dg)$ such that for every $1 \leqq j \leqq r$,

$$\left\| h_j - \int_G h \circ \alpha_g \cdot \gamma_j(g) \cdot \frac{d\mu\alpha_g}{d\mu}\, dg \right\|_1 \leqq \varepsilon,$$

where $d\mu\alpha_g / d\mu$ denotes the Radon–Nikodym derivative of $\alpha_g \cdot \mu$ with respect to μ. We say also that α is AT, or, when $G = \mathbf{Z}$ and the action is given by a single transformation $f = \alpha_1$, we say that f is AT. A flow built under a constant ceiling function is AT if and only if the base transformation is AT [7].

We now state some results on AT transformations and flows.

THEOREM 4.2. [7] *If $f \in \mathrm{Aut}(X, \mathcal{S}, \mu)$ is* AT, *then f is ergodic.*

THEOREM 4.3. [7] *If W^* is a Krieger factor which is* ITPFI, *then the associated flow of weights is* AT.

COROLLARY. 4.4. *If f is an ergodic automorphism of $(X, \mathcal{S}, \mu)$ which is* ITPFI, *then its associated flow is* AT.

THEOREM 4.5. [7] *If f is a finite measure-preserving transformation which is* AT, *then f has zero entropy.*

Our task is now a simple one. We consider the diffeomorphisms of $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ given by the matrices

$$U_n = \begin{pmatrix} n+1 & 1 \\ n & 1 \end{pmatrix};$$

that is, $U_n(y_1, y_2) = ((n+1)y_1 + y_2, ny_1 + y_2) \pmod 1$ for every integer $n \geqq 1$, and for every $(y_1, y_2) \in T^2$. Each U_n gives an ergodic, measure-preserving group automorphism of the torus isomorphic to a Bernoulli shift [13]. Since the entropy of U_n is $\log((n + 2 + \sqrt{n(n+4)})/2)$, then Theorem 4.5 implies that U_n is not AT. We now take the suspension flow of U_n for each n, and we can easily check that we obtain a countable family of aperiodic, conservative, ergodic flows which are all mutually non-isomorphic.

The following result is now easily proved.

THEOREM 4.6. *There exists a C^∞ diffeomorphism of T^4 which is non*-ITPFI.

PROOF. Let $f \in \mathrm{Diff}^\infty(T^1)$ be a type III_1 diffeomorphism. Let $\{U_s\}_{s\in\mathbf{R}}$ be the flow defined as above for $n = 1$, i.e., $U : T^2 \to T^2$ is given by $U(y_1, y_2) = (2y_1 + y_2, y_1 + y_2) \pmod 1$, and $\{U_s\}$ is the suspension flow of U on T^3.

By Corollaries 3.4 and 4.4 the map $K : T^4 \to T^4$ defined by:

$$K(x, \bar{y}) = (fx, U_{\log(dmf^{-1}(x)/dm)}(\bar{y})) \qquad \text{for every } x \in T^1, \quad \bar{y} \in T^3$$

is a diffeomorphism which is non-ITPFI.

COROLLARY. 4.7. *There exists a countably infinite family of weak equivalence classes of non-ITPFI diffeomorphisms of T^4.*

§5. Diffeomorphisms of higher dimensional manifolds

In this section we use methods from [10] and [11] to extend our construction to higher dimensions. The following lemma is necessary to generalize the construction.

LEMMA 5.1. *Suppose $K : T^4 \to T^4$ is defined as in Theorem 4.6. Let $K_\psi : T^4 \times \mathbf{R} \to T^4 \times \mathbf{R}$ be defined by:*

$$K_\psi(x, \bar{y}, t) = (K(x, \bar{y}), t + \psi(x, \bar{y}))$$

for every $(x, \bar{y}) \in T^4$, $t \in \mathbf{R}$, and $\psi \in C^\infty(T^4, \mathbf{R})$; suppose also that K_ψ is ergodic with respect to Lebesgue measure on $T^4 \times \mathbf{R}$ and is of type III_0. Then K_ψ is non-ITPFI.

PROOF. The idea of the proof is to show that K_ψ is weakly equivalent to K, and is therefore non-ITPFI.

We first remark that by Corollary 3.4, K is weakly equivalent to:

$$\bar{K}(x, \bar{y}) = (fx, U_{\log(\bar{m}f^{-1}(x)/d\bar{m})}(\bar{y}))$$

where $\bar{m} \sim m$ is a strictly f-admissible measure on T^1. We then claim that $\bar{K}$ is weakly equivalent to $\bar{K}_\psi$, which is defined on $T^4 \times \mathbf{R}$ by:

$$\bar{K}_\psi(x, \bar{y}, t) = (\bar{K}(x, \bar{y}), t + \psi(x, \bar{y})).$$

It follows that $\bar{K}_\psi$ has the same associated ergodic flow, $\{U_s\}$, as $\bar{K}$. (The associated factor map for $\bar{K}_\psi$ is $\bar{\varphi} : T^1 \times T^3 \times \mathbf{R} \times \mathbf{R} \to T^3$ given by $\bar{\varphi}(x, \bar{y}, t, s) = U_s(\bar{y})$.) Another application of Corollary 3.4 shows that $\bar{K}_\psi$ is weakly equivalent to K_ψ. The result follows from the transitivity of weak equivalence.

To see that Lemma 5.1 is not vacuous, we apply a theorem from [10]. Let $(X, \mathcal{S}, \mu)$ denote a smooth connected paracompact manifold with μ a C^∞

probability measure on X. Let $g \in \mathrm{Diff}^\infty(X)$ be an ergodic diffeomorphism. We define the set

$$\mathscr{C} = \mathrm{cl}\{\psi \in C^\infty(X,\mathbf{R}) \mid \psi = \eta - \eta \circ g \text{ for some Borel map } \eta : X \to \mathbf{R}\},$$

where cl denotes the closure taken with respect to the C^∞ topology in $C^\infty(X,\mathbf{R})$. The next theorem states that there are many functions in $\mathscr{C}$ (in the Baire category sense) which give ergodic extensions if g is of type III_0.

THEOREM 5.2. *Suppose that* $g \in \mathrm{Diff}^\infty(X)$ *is an ergodic type* III_0 *diffeomorphism. Then the set*

$$\mathscr{C}_0 = \{\psi \in \mathscr{C} \mid (z,t) \mapsto (gz, t + \psi(z)) \ \forall z \in X, t \in \mathbf{R}, \text{ is of type } \mathrm{III}_0\}$$

is a dense G_δ *in* $\mathscr{C}$.

We use this to prove the next theorem.

THEOREM 5.3. *There exists a diffeomorphism of* $T^4 \times \mathbf{R}^p$ *for every* $p \geqq 0$, *which is* C^∞ *and non*-ITPFI.

PROOF. We use induction on p. We start with $K \in \mathrm{Diff}^\infty(T^4)$ defined in Theorem 4.6. For $p = 1$, the theorem is true by Lemma 5.1 and Theorem 5.2. Assume the theorem is true for $p = j$. Then suppose that $K_j \in \mathrm{Diff}^\infty(T^4 \times \mathbf{R}^j)$ is a non-ITPFI diffeomorphism. By Theorem 5.2 there is at least one function $\psi : T^4 \times \mathbf{R}^j \to \mathbf{R}$ such that $(z_j, t) \mapsto (K_j(z_j), t + \psi(z_j))$ $\forall z_j \in T^4 \times \mathbf{R}^j$, $t \in \mathbf{R}$, is of type III_0. Then by Lemma 5.2 this map is a non-ITPFI diffeomorphism of $T^4 \times \mathbf{R}^{j+1}$.

Finally, to extend our result to arbitrary manifolds of dimension $\geqq 6$ we apply the following lemmas.

LEMMA 5.4. [11] *Let* X *be a* p*-dimensional* C^∞ *paracompact connected manifold and* μ *a* C^∞ *measure on* X. *Then there exists an open set* $V \subset X$, *diffeomorphic to* $\mathbf{R}^p$ *and satisfying* $\mu(X - V) = 0$.

LEMMA 5.5. [11] *If* $p \geqq 6$, *there exists an open set* W *of* $\mathbf{R}^p$ *diffeomorphic to* $T^5 \times \mathbf{R}^{p-5}$ *such that* $m(\mathbf{R}^p - W) = 0$.

LEMMA 5.6. *Let* $K_0 \in \mathrm{Diff}(T^4)$ *denote the ergodic non*-ITPFI *diffeomorphism* (K) *defined in Theorem* 4.6, *and by* K_j, $j \geqq 1$, *we will denote a diffeomorphism of* $T^4 \times \mathbf{R}^j$ *of the form*:

$$(z, t_1, t_2, \cdots, t_j) \mapsto (K_0(z), t_1 + \psi_1(z), \cdots, t_j + \psi_j(z, t_1, \cdots, t_{j-1}))$$

for every $z \in T^4$, $t_i \in \mathbf{R}$, $1 \leqq i \leqq j$. *If* $F_s^j : T^4 \times \mathbf{R}^j \times T^1 \circlearrowleft$ *denotes the suspension flow of* K_j, *then for* m-a.e. $s_0 \in \mathbf{R}$, *and for every* $j \geqq 0$, $F_{s_0}^j$ *is a non*-ITPFI *diffeomorphism.*

Proof. Since K_j is ergodic, then for almost every $s_0 \in \mathbf{R}$, $F_{s_0}^j$ is ergodic [11]. Using the same argument as in the proof of Lemma 5.1, we see that K_j and $F_{s_0}^j$ have the same ergodic flow associated to them so they are weakly equivalent, hence $F_{s_0}^j$ is non-ITPFI.

Lemma 5.7. [10, 11] *Let* W *be an open set of* $\mathbf{R}^p$, *and let* F_s *denote a* C^∞ *flow of type* III *on* W. *Let* χ *be the infinitesimal generator of* F_s, *i.e.,* χ *is defined by*:

$$\frac{\partial F_s}{\partial s}(w)\Big|_{s=0} = \chi \circ F_s(w) \qquad \forall w \in W.$$

We define $\varphi \in C^\infty(W, \mathbf{R})$, $\varphi > 0$ *such that the vector field* $\varphi\chi$ *is globally integrable and defines a flow* G_s. *Then* G_s *is weakly equivalent to* F_s.

Theorem 5.8. *There exists a* C^∞ *non*-ITPFI *diffeomorphism on every connected, paracompact manifold of dimension* $\geqq 6$.

Proof. By Lemmas 5.4 and 5.5 there exists an open set $W \subset X$ of full measure and such that W is diffeomorphic to $T^5 \times \mathbf{R}^{p-5}$ (where X is of dimension $p \geqq 6$). By Theorem 5.3 there exists a C^∞ non-ITPFI diffeomorphism of $T^4 \times \mathbf{R}^{p-5}$; we denote it by K_{p-5}. We then take the suspension flow of K_{p-5}, denoted F_s^{p-5}, as in Lemma 5.6. Suppose that χ^{p-5} denotes the infinitesimal generator of F_s^{p-5}. We now define $\varphi \in C^\infty(X, \mathbf{R})$ such that $\varphi > 0$ on W, $\varphi = 0$ on $X - W$, and such that the vector field

$$Y(x) = \begin{cases} \varphi(x)\chi^{p-5}(x), & \text{if } x \in W \\ 0, & \text{if } x \in X - W \end{cases}$$

is C^∞ on X and globally integrable, thus defining a flow G_s^{p-5} on X. By Lemma 5.7, G_s^{p-5} is weakly equivalent to F_s^{p-5}. Then Lemma 5.6 implies that for m-a.e. $s_0 \in \mathbf{R}$, $G_{s_0}^{p-5}$ is a non-ITPFI diffeomorphism of X.

References

1. H. Araki and E. J. Woods, *A classification of factors*, Publ. Res. Inst. Math. Sci., Ser. A **4** (1968), 51–130.

2. L. K. Arnold, *On σ-finite invariant measures*, Z. Wahrscheinlichkeitstheor. Verw. Geb. **9** (1968), 85–97.

3. A. Connes, *Une classification des facteurs de type III*, Ann. Sci. Ec. Norm. Super. **6** (1973), 133–252.

4. A. Connes, *On hyperfinite factors of type III_0 and Krieger's factors*, J. Funct. Anal. **18** (1975), 318–327.

5. A. Connes and M. Takesaki, *The flow of weights on factors of type III*, Tohoku Math. J. **29** (1977), 473–575.

6. A. Connes and E. J. Woods, *A construction of approximately finite-dimensional non-ITPFI factors*, Can. Math. Bull. **23** (1980), 227–230.

7. A. Connes and E. J. Woods, *Approximately transitive flows and ITPFI factors*, to appear.

8. T. Hamachi and M. Osikawa, *Ergodic groups of automorphisms and Krieger's theorems*, preprint (1980).

9. T. Hamachi, Y. Oka and M. Osikawa, *Flows associated with ergodic non-singular transformation groups*, Publ. Res. Inst. Math. Sci. **11** (1975), 31–50.

10. J. Hawkins, *Smooth Type III diffeomorphisms of manifolds*, Trans. Am. Math. Soc., to appear.

11. M. R. Herman, *Construction de difféomorphismes ergodiques*, preprint (1979).

12. M. R. Herman, *Sur la conjugaison differentiable des diffeomorphismes du cercle a des rotations*, Publ. I.H.E.S. No. 49 (1979), 5–234.

13. Y. Katznelson, *Ergodic automorphisms of T^n are Bernoulli shifts*, Isr. J. Math. **10** (1971), 186–195.

14. Y. Katznelson, *Sigma-finite invariant measures for smooth mappings of the circle*, J. Analyse Math. **31** (1977), 1–18.

15. Y. Katznelson, *The action of diffeomorphisms of the circle on Lebesgue measure*, J. Analyse Math. **36** (1979), 156–166.

16. U. Krengel, *Darstellungssätze für Strömungen und Halbströmungen II*, Math. Ann. **182** (1969), 1–39.

17. W. Krieger, *On non-singular transformations of a measure space I*, Z. Wahrscheinlichkeitstheor. Verw. Geb. **11** (1969), 83–97.

18. W. Krieger, *On non-singular transformations of a measure space II*, Z. Wahrscheinlichkeitstheor. Verw. Geb. **11** (1969), 98–119.

19. W. Krieger, *On the Araki–Woods asymptotic ratio set and non-singular transformations of a measure space*, Lecture notes in Math. **160**, Springer-Verlag, 1970, pp. 158–177.

20. W. Krieger, *On constructing non-* isomorphic hyperfinite factors of type III*, J. Funct. Anal. **6** (1970), 97–109.

21. W. Krieger, *On a class of hyperfinite factors that arise from null-recurrent Markov chains*, J. Funct. Anal. **7** (1971), 27–42.

22. W. Krieger, *On the infinite product construction of non-singular transformations of a measure space*, Invent. Math. **15** (1972), 144–163.

23. W. Krieger, *On ergodic flows and isomorphism of factors*, Math. Ann. **223** (1976), 19–70.

24. I. Kubo, *Quasi-flows*, Nagoya Math. J. **35** (1969), 1–30.

25. J. von Neumann, *On rings of operators III*, Ann. of Math. **41** (1940), 94–161.

26. D. S. Ornstein, *On invariant measures*, Bull. Am. Math. Soc. **66** (1960), 297–300.

27. C. E. Sutherland, *Notes on orbit equivalence*; *Krieger's theorem*, Lecture note series no. 23, Univ. i Oslo, 1976.

28. E. J. Woods, *ITPFI factors — a survey*, Queen's University, preprint (1980).

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